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On the velocity-dependence of image forces

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Abstract
Classical theory yields an exact expression for the Lorentz force $\mathbf{f}(\zeta, \mathbf{u})$ on a point charge Q which has third coordinate ζ, and which has always moved with strictly constant velocity $\mathbf{u}$ in a halfspace bounded by a perfectly reflecting xy plane. On its own, $\mathbf{f}$ would produce an acceleration directed towards the surface, smaller for any nonzero than for vanishing $\mathbf{u}$. It is compared, to orders u^2/c^2 and Q^2, with quantum-perturbative predictions for an otherwise free particle near variously modelled imperfect reflectors. For large ζ and/or high reflectivity, the quantum results are dominated by components free of $\hbar$, and we reason that such components should agree with the corresponding classical perfect-reflector results. This test is applied to several theories: some pass and some fail.

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1. Introduction and conclusions

1.1. Motivation and preview

Quantum mechanics occasionally makes predictions that are free of Planck's constant, and the writer believes that these must agree with the corresponding predictions of classical physics. Nonrelativistic examples include the Coulomb cross-section at all energies and angles, the isotropic high-energy cross-section for hard-sphere scattering outside the forward diffraction peak, and specific heats at high temperatures. Here we shall apply this criterion to the force on a point charge near a reflecting surface, a poor relation of the famous Casimir–Polder force on an atom. The classical predictions obtain in the limits of perfect reflection, or large distance, or both.

Presently this introduction will spell out Maxwell's equations and the pertinent boundary conditions for perfect reflection, stressing, by hindsight, that the problem is fully determined ('well posed') without reference to physics on the far side of the surface. Section 2 determines, exactly, the classical fields due to a point charge Q constrained to move with strictly constant velocity $\mathbf{u}$ near and always to one side of such a perfectly reflecting plane. The central result is equation (2.7) for the acceleration $\mathbf{a}$ that they would produce in the absence of the constraint, plus the perhaps surprising realization of just how tightly $\mathbf{a}$ is circumscribed by symmetries alone. Section 3 cites, still for perfect reflection, the conventionally-derived quantum-perturbative result, concentrated as usual into an effective Hamiltonian (3.1) accurate to orders Q^2 and $(u/c)^2$; derives its implications via Hamilton's equations; and verifies that they tally with the corresponding results from section 2.

Finally, section 4 cites quantum results variously derived for several different models of reflectors, works out their consequences, and checks whether in the appropriate regimes they too agree with classical physics. Those favoured by the writer do; some favoured by some of his friends do not. Regarding plasmas, no attempt has yet been made to trace the precise technical roots of the discrepancies. Regarding a nondispersive dielectric half-space with real refractive index n, the discrepancies stem from a primary dependence of convective fields on nu/c, additionally to their more explicit variation with n and with u/c separately.

1.2. Perfect reflection classically

We consider a point particle with charge Q and mass m, which hitherto (i.e. for all times $t > -\infty$) has followed, in the $z > 0$ half-space, a prescribed constant-velocity trajectory $\boldsymbol{\rho}(t) \equiv (\boldsymbol{\rho}_\parallel(t), \zeta(t)) \equiv (\xi(t), \eta(t), \zeta(t))$, $\mathrm{d}\boldsymbol{\rho}/\mathrm{d}t \equiv \mathbf{u} = (\mathbf{u}_\parallel, u_3)$. Evidently u_3 cannot be positive. We choose

$$\boldsymbol{\rho}(0) = (\mathbf{0}, \zeta), \tag{1.1}$$

subject to the strict inequality $\zeta > 0$. The half-space is bounded by the perfectly reflecting xy plane. Thus[1], with $\mathbf{r}_\parallel \equiv (x, y)$ and $z > 0$,

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} - \partial \mathbf{B}/c\partial t = 0, \tag{1.2}$$

$$\nabla \cdot \mathbf{E} = 4\pi Q\delta(\mathbf{r} - \boldsymbol{\rho}(t)), \qquad \nabla \times \mathbf{B} - \partial \mathbf{E}/c\partial t = 4\pi Q(\mathbf{u}/c)\delta(\mathbf{r} - \boldsymbol{\rho}(t)); \tag{1.3}$$

$$\lim_{z\to 0+} \mathbf{E}_\parallel(\mathbf{r}_\parallel, z) = 0 \quad \Rightarrow \quad \lim_{z\to 0+} \nabla_\parallel \cdot \mathbf{E}_\parallel(\mathbf{r}_\parallel, z) = 0, \qquad \lim_{z\to 0+} [\nabla \times \mathbf{E}(\mathbf{r}_\parallel, z)]_3 = 0. \tag{1.4}$$

In view of (1.4) the first of equations (1.3) and the second of equations (1.2), respectively, entail

$$\lim_{z\to 0+} \partial E_3(\mathbf{r}_\parallel, z)/\partial z = 0, \qquad \lim_{z\to 0+} \partial B_3(\mathbf{r}_\parallel, z)/\partial t = 0. \tag{1.5}$$

The second condition evidently fails to constrain time-independent contributions to $B_3(z \to 0+)$, which in general need vanish only on superconductors exhibiting the Meissner effect. Into all other types of more or less realizable media the B fields of uniformly moving charges do penetrate, as they must for combinations of them to realize the Ampèrian field of a steady current parallel to the surface. To describe such penetration into Ohmic conductors is known to be a very delicate problem (Boyer 1974, 1999), the more so because of incompatibilities between the limits of vanishing speed and of perfect conduction; and because expressing velocity-fields as linear superpositions of radiative normal modes is bedevilled by the fact that for all modes with nonzero frequency (1.5) evidently does ensure that their amplitudes B_3 vanish on the surface. Hence it is fortunate that we are concerned with charges having finite velocities and thereby time-varying fields, which the perfect-reflection limit does eventually exclude from all materials; accordingly, in that limit (1.5) can indeed be sharpened to

$$\lim_{z\to 0+} \partial E_3(\mathbf{r}_\parallel, z)/\partial z = 0, \qquad \lim_{z\to 0+} B_3(\mathbf{r}_\parallel, z) = 0, \tag{1.6}$$

which are the boundary conditions universally adopted by analyses of normal modes.

[1] We use unrationalized Gaussian units: the fine-structure constant for instance would read $e^2/\hbar c \simeq 1/137$.

We stress that our boundary conditions feature only the fields at $z \geqslant 0$. Provided these conditions are satisfied, the physics in the right-hand half-space is uniquely determined, regardless of what happens to the left. Furthermore, this and the next section make a point of working only with the fields, so that the results are automatically gauge-invariant: potentials will enter (implicitly) only with the Hamiltonians considered in sections 3 and 4.

2. The Lorentzian acceleration

At $t = 0$ the charge experiences the Lorentz force

$$\mathbf{f} = Q\{\mathbf{E}(\mathbf{0}, \zeta) + \mathbf{u} \times \mathbf{B}(\mathbf{0}, \zeta)/c\}. \tag{2.1}$$

If the constraints confining the particle to the prescribed trajectory $\boldsymbol{\rho}(t)$ were suddenly switched off, $\mathbf{f}$ would produce an acceleration (Landau and Lifshitz 1975, section 17)

$$\mathbf{a} = \frac{Q}{m}\sqrt{1 - u^2/c^2}\left\{\mathbf{E} - \frac{1}{c^2}\mathbf{u}(\mathbf{u}\cdot\mathbf{E}) + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right\}. \tag{2.2}$$

This section aims finds $\mathbf{f}$ and thence $\mathbf{a}$. It does so without representing the fields as integrals over normal modes, partly to avoid worries about whether it is legitimate to implement any limits in question already on the integrands.

The retarded fields of a point charge moving uniformly in unbounded space are well known: call them $\mathbf{E}_0, \mathbf{B}_0$. To satisfy the Maxwell equations (1.2), (1.3) subject to the boundary conditions (1.4), (1.6), one need merely add the corresponding fields $\tilde{\mathbf{E}}, \tilde{\mathbf{B}}$ generated by a particle of charge $\tilde{Q}$ following the mirror trajectory $\tilde{\boldsymbol{\rho}}$, where

$$\tilde{Q} = -Q, \qquad \tilde{\boldsymbol{\rho}}(t) \equiv (\boldsymbol{\rho}_{\|}(t), -\zeta(t)), \qquad \mathrm{d}\tilde{\boldsymbol{\rho}}/\mathrm{d}t \equiv \tilde{\mathbf{u}}(t) = (\mathbf{u}_{\|}(t), -u_3(t)). \tag{2.3}$$

Thus[2] $\mathbf{E} = \mathbf{E}_0 + \tilde{\mathbf{E}}$, and likewise for $\mathbf{B}$. However, we wish to exclude from $\mathbf{f}$ the force that the particle would exert on itself in the absence of any reflectors; in other words we drop the fields $\mathbf{E}_0, \mathbf{B}_0$, and find $\mathbf{a}$ from (2.2) after replacing $(\mathbf{E}, \mathbf{B}) \rightarrow (\tilde{\mathbf{E}}, \tilde{\mathbf{B}})$ there. Accordingly (Landau and Lifshitz 1975, section 38)

$$\mathbf{E} \rightarrow \frac{\tilde{Q}\mathbf{R}(1 - \tilde{u}^2/c^2)}{R^3[1 - (\tilde{u}/c)^2 \sin^2\tilde{\theta}]^{3/2}}, \qquad \mathbf{B} \rightarrow \tilde{\mathbf{u}} \times \mathbf{E}/c, \tag{2.4}$$

$$\mathbf{R} \equiv \boldsymbol{\rho} - \tilde{\boldsymbol{\rho}} = \hat{\mathbf{z}}2\zeta, \qquad \cos\tilde{\theta} \equiv \frac{\mathbf{R}\cdot\tilde{\mathbf{u}}}{R\tilde{u}} = \frac{\tilde{u}_3}{\tilde{u}} = -\frac{u_3}{u}. \tag{2.5}$$

Hats specify unit vectors. It follows straightforwardly that

$$\mathbf{E} = -\hat{\mathbf{z}}\mathcal{E}, \qquad \mathbf{B} = (\hat{\mathbf{z}} \times \mathbf{u}/c)\mathcal{E}, \qquad \mathcal{E} \equiv \frac{Q}{4\zeta^2}\cdot\frac{[1 - u^2/c^2]}{\left[1 - u_{\|}^2/c^2\right]^{3/2}}. \tag{2.6}$$

On substitution into (2.2) there is some cancellation between the electric and magnetic contributions to the velocity-dependent parts, and eventually one obtains

$$\mathbf{a} = -\hat{\mathbf{z}}\frac{Q^2}{4\zeta^2 m}\cdot\frac{[1 - u^2/c^2]^{5/2}}{\left[1 - u_{\|}^2/c^2\right]^{3/2}} = -\hat{\mathbf{z}}\frac{Q^2}{4\zeta^2 m}\left\{1 - u_{\|}^2/c^2 - 5u_3^2/2c^2 + \mathcal{O}(u/c)^4\right\}. \tag{2.7}$$

Note that nonzero $\mathbf{u}$ always diminishes the acceleration towards the boundary: in this sense, all velocity-dependent corrections are repulsive.

[2] Such simplicity stems from symmetries attending $\tilde{Q} = -Q$, which are peculiar to perfect reflection. For reflection of any other kind the corresponding calculations are far more complicated.

It is worth observing that symmetries alone force $\mathbf{a}_\parallel$ to vanish. If $\mathbf{u}'_\parallel = \mathbf{0}$ relative to the rest-frame S' of the medium, then $\mathbf{a}'_\parallel = \mathbf{0}$ simply by parity, whatever the reflectivity. But if the material reflects perfectly, then the boundary conditions reduce to (1.4), (1.6) regardless of its velocity parallel to the surface, so that they as well as the field equations become invariant under surface-parallel Lorentz boosts. Let the frame S have velocity $-\mathbf{u}_\parallel$ relative to S', so that the particle velocity relative to S is $\mathbf{u}_\parallel + \hat{\mathbf{z}}u_3$. Then, given $\mathbf{u}'_\parallel = \mathbf{0} = \mathbf{a}'_\parallel$, the Lorentz transformation of accelerations from S' to S immediately prescribes $\mathbf{a}_\parallel = \mathbf{0}$, as in (2.7). Likewise one can verify that such boosts fully determine how a_3 varies with $u_\parallel$.

The next section will compare the terms of order u^2/c^2 with results from perturbation theory. The textbooks discuss how restricting $\mathbf{f}$ to this order validates it with $\mathbf{u}$ interpreted as the instantaneous velocity, liberating one from the assumption that $\mathbf{u}$ has been constant at all times up to the present.

3. Quantum perturbation theory

3.1. An effective Hamiltonian

The standard approach to the velocity dependence is indirect: quantize the Maxwell field in the half-space, using the pseudo-Coulomb gauge for the vector potential $\mathbf{A}$, with $\nabla \cdot \mathbf{A} = 0$ but $A_3(\mathbf{r}_\parallel, 0+) \neq 0$ (Barton 1977, 2007); couple particle to field through $H_{\text{int}} = -Q^2/4\zeta - (Q/mc)\mathbf{A}(\boldsymbol{\rho}) \cdot p + (Q^2/2mc^2)\mathbf{A}^2(\boldsymbol{\rho})$, with $\boldsymbol{\rho}$, $\mathbf{p}$ now the canonical variables of the particle; and for a particle localized quasi-classically in both $\boldsymbol{\rho}$ and $\mathbf{p}$ write down the ground-state energy shift second-order in $-(Q/mc)\mathbf{A}(\boldsymbol{\rho}) \cdot \mathbf{p}$. For simplicity, we shall designate such shifts, and other allied expressions, as 'of order u^2/c^2': although the difference between $\mathbf{p}/mc$ and $\mathbf{u}/c$ will presently be seen to matter a great deal, it does not prejudice the usefulness of this shorthand. In an equally obvious sense, we shall speak of and restrict ourselves to 'terms of order Q^2'. To these orders, one can and does adopt the so-called no-recoil approximation to the energy denominators, dropping particle energies and keeping only the energy of the virtual photon.

Crucially, the result is free of $\hbar$. Adjoining it to the kinetic energy plus the unretarded image potential, one obtains the quantally derived but in the event purely classical effective Hamiltonian:

$$H = p^2/2m - Q^2/4\zeta + \{-p^4/8m^3c^2 + \Delta H\}, \qquad \Delta H \equiv (Q^2/m^2c^2\zeta)\big[p_\parallel^2/8 - p_3^2/4\big]. \tag{3.1}$$

The braces combine ΔH with the first relativistic correction to the kinetic energy, needed for consistency to order $(u/c)^2$. (One can easily check that making the latter gauge-invariant would have no effect to the orders to which we are working.) The interaction ΔH has been doctored to ensure that it vanishes as $\zeta \to \infty$, and terms proportional to $\delta(\zeta)$ have been dropped. The leading correction excluded from H is the typically quantal vacuum expectation-value $(Q^2/2mc^2)\{1 - \lim_{\zeta\to\infty}\}\langle\mathbf{A}^2(\boldsymbol{\rho})\rangle = Q^2\hbar/4\pi mc\zeta^2$. It measures the kinetic energy of the random motion driven by the zero-point oscillations of the quantized field, and is smaller than ΔH when $\zeta \gg (\hbar/mc)(c^2/u^2)$.

An alternative way to establish H simply writes down the $\mathcal{O}(u/c)^2$ Darwin Hamiltonian for two particles a and b interacting electromagnetically (Landau and Lifshitz 1975, equation (65.8)), and then adapts it to our scenario by identifying a as the particle we actually have, and assimilating b to the image. Accordingly one drops the kinetic energy of b; replaces the separation R_{ab} by 2ζ and $\hat{\mathbf{R}}_{ab}$ by $\hat{\mathbf{z}}$; in the interaction replaces $\mathbf{p}_b \to (\mathbf{p}_{a\parallel} - \hat{\mathbf{z}}p_{a3})$; and finally multiplies the entire interaction by the familiar image factor $1/2$.

Our object is to compare the velocity-dependent accelerations from (2.7) with those that follow canonically from H. Since the conditions that validate (2.7) also validate ΔH (though not vice versa) the two expressions must agree[3]. The introduction has already explained that the main point of the exercise is diagnostic: some applications are discussed in section 4. As to verifying ΔH through the motion of ions near metal surfaces, the prospects are poor, because in unmodified form it applies only at large ζ, where the effects are weak, while at small ζ it is modified by Ohmic friction, and by the dispersive effects of finite plasma frequencies. The most one can say is that expectation values of ΔH turn up in some subdominant parts of mirror-induced *atomic* energy shifts.

3.2. The acceleration via Hamilton's equations

Recall that H in (3.1), and therefore its consequences, are warranted only to orders Q^2 and u^2/c^2. From H, we obtain

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = u_3 = \frac{\partial H}{\partial p_3} = \frac{p_3}{m}\left[1 - \frac{p_\parallel^2 + p_3^2}{2m^2c^2} - \frac{Q^2}{2mc^2\zeta}\right], \tag{3.2}$$

$$\frac{\mathrm{d}\boldsymbol{\rho}_\parallel}{\mathrm{d}t} = \mathbf{u}_\parallel = \frac{\partial H}{\partial \mathbf{p}_\parallel} = \frac{\mathbf{p}_\parallel}{m}\left[1 - \frac{p_\parallel^2 + p_3^2}{2m^2c^2} + \frac{Q^2}{4m\zeta}\right]; \tag{3.3}$$

$$\mathrm{d}p_3/\mathrm{d}t = -\partial H/\partial\zeta = -(Q^2/4\zeta^2)\left[1 - p_\parallel^2/2m^2c^2 + p_3^2/m^2c^2\right], \tag{3.4}$$

$$\mathrm{d}\mathbf{p}_\parallel/\mathrm{d}t = -\partial H/\partial\boldsymbol{\rho}_\parallel = 0. \tag{3.5}$$

Now differentiate (3.3), use (3.5), substitute from (3.4), and rearrange:

$$\begin{aligned} md^2\boldsymbol{\rho}_\parallel/d^2t &= \mathbf{p}_\parallel\{-(p_3/m^2c^2)\,\mathrm{d}p_3/\mathrm{d}t - (Q^2/4mc^2\zeta)\,\mathrm{d}\zeta/\mathrm{d}t\} \\ &= \mathbf{p}_\parallel\left\{\frac{p_3}{m^2c^2}\left[\frac{Q^2}{4\zeta^2} + \frac{Q^2}{8m^2c^2\zeta^2}\left(p_\parallel^2 - 2p_3^2\right)\right] - \frac{Q^2}{4m^2c^2\zeta^2}\cdot\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right\} = 0. \end{aligned} \tag{3.6}$$

The final equality follows (i) because to overall order u^2/c^2 the second term inside the square brackets drops out; and (ii) because to overall order Q^2 we may in the last term replace $\mathrm{d}\zeta/\mathrm{d}t \to p_3/m$.

Similarly, differentiate (3.2), again use (3.5), substitute from (3.2) itself, and rearrange:

$$m\frac{\mathrm{d}^2\zeta}{\mathrm{d}t^2} = -\frac{Q^2}{4\zeta^2}\left[1 - \frac{\left(p_\parallel^2 - 2p_3^2\right)}{2m^2c^2\zeta^2}\right]\left[1 - \frac{\left(p_\parallel^2 + 3p_3^2\right)}{2m^2c^2} - \frac{Q^2}{2mc^2\zeta}\right] + \frac{Q^2}{2mc^2\zeta^2}p_3\frac{\mathrm{d}\zeta}{\mathrm{d}t}.$$

To overall order Q^2 we must drop the last term inside the second pair of square brackets, and may replace all $\mathbf{p} \to m\mathbf{u}$; and from the product of the two pairs of square brackets we must drop the terms of overall order u^4/c^4. A final rearrangement then reproduces the Lorentzian result (2.7).

We stress that not only the precise magnitudes but the entire pattern of the accelerations depend critically on the kinetic-energy correction $-p^4/8m^3c^2$ in H. Without it the acceleration would read

$$\mathbf{a}_\parallel = -(Q^2/4\zeta^2m)u_3\mathbf{u}_\parallel/c^2, \qquad a_3 = (Q^2/4\zeta^2m)\left[1 - u_\parallel^2/2c^2 - u_3^2/c^2\right]: \quad \text{wrong.}$$

[3] Prima facie this is paradoxical: the velocity-dependent part of (2.7) is purely repulsive, whereas the $p_\parallel^2$—and p_3^2—proportional parts of ΔH have opposite signs, whence one might well have expected their effects to turn out repulsive and attractive respectively.

For other comparisons, it will prove worth recording that if one replaced (3.1) by the less specific Hamiltonian

$$H' \equiv p^2/2m - \alpha Q^2/4\zeta - p^4/8m^3c^2 + (Q^2/m^2c^2\zeta)\big[\lambda p_\parallel^2 + \mu p_3^2\big] \tag{3.7}$$

one would find

$$\mathbf{a}' = \frac{Q^2}{\zeta^2 m}\left\{\frac{\mathbf{u}_\parallel u_3}{c^2}\left(\frac{\alpha}{4} - 2\lambda\right) + \hat{\mathbf{z}}\left[-\frac{\alpha}{4} + \frac{u_\parallel^2}{c^2}\left(\frac{\alpha}{8} + \lambda\right) + \frac{u_3^2}{c^2}\left(\frac{3\alpha}{8} - \mu\right)\right]\right\}. \tag{3.8}$$

4. Diagnostics

The interaction ΔH in (3.1), operative in the half-space $z > 0$, emerges not only when the xy plane is treated as perfectly reflecting from the start, but also as an appropriate limit in some slightly less idealized models for the boundary.

(i) First, the half-space $z < 0$ may be modelled as a plasma with the bulk dielectric response function $\varepsilon = 1 - \omega_p^2/\omega^2$ at frequency ω, perfect reflection being approached as the plasma-frequency ω_p tends to infinity. This model yields the momentum-dependent part of the interaction in the form (Barton 1977)

$$(Q^2/m^2c^2\zeta)\big[\Phi^{(\parallel)}(s)p_\parallel^2 + \Phi^{(3)}(s)p_3^2\big], \qquad s \equiv 2\omega_p\zeta/c, \tag{4.1}$$

with dimensionless functions[4] $\Phi(s)$ related to Bessels. The dimensional limits $\omega_p \to \infty$ and $\zeta \to \infty$ evidently commute with each other, but not with the so-called nonrelativistic/nonretarded limit $c \to \infty$; and it is $s \to \infty$ that reproduces (3.1). By contrast, as $s \to 0$ one finds $-\big(Q^2/8m^2\omega_p^2\zeta^3\big)\big[p_\parallel^2 + 2p_3^2\big]$; however, with decreasing s the model eventually fails through its disregard of Debye-type wave-number cutoffs and of spatial dispersion.

(ii) Second, the xy plane may be modelled as an infinitesimally thin but finitely reflecting 2D plasma sheet (Barton 2005)[5], roughly mimicking a single base plane from graphite. Its optical properties are governed by the dimensional parameter $q_p \equiv 2\pi ne^2/\bar{\mu}c^2$, with $ne, n\bar{\mu}$ the equilibrium charge and mass of the mobile charge carriers per unit area. The analogue of (4.1) then replaces the $\Phi(s)$ by dimensionless functions $\Psi(\sigma \equiv 2q_p\zeta)$, related to exponential integrals. Their various limits are subject to comments analogous to those about Φ's. Here too $\sigma \to \infty$ reproduces (3.1); by contrast, $\sigma \to 0$ produces $-(Q^2\bar{\mu}/32\pi ne^2m^2\zeta^2)\big[p_\parallel^2 + 2p_3^2\big]$, with the model eventually failing for the same reasons as the 3D model fails.

(iii) A different theory for the same models has been proposed by Bordag (2004, 2007: cited as MB). In the perfect-reflection and/or large-distance limits it agrees with (3.1) when the boundary is backed by a 3D plasma occupying the half-space $z < 0$; but when the boundary is constituted by the 2D sheet described above, MB derives a different interaction, replacing our ΔH with

$$\Delta H|_{MB} = (Q^2/m^2c^2\zeta)\big[-p_\parallel^2/8 - p_3^2/4\big], \tag{4.2}$$

while nevertheless admitting the boundary conditions (1.4), (1.6) in the perfect-reflection limit. Through (3.7), (3.8), equation (4.2) predicts accelerations

[4] Though ΔH is restricted to second order in u/c, the model applies to all orders in $s \equiv 2\omega_p\zeta/c$. That is one reason never to speak simply of 'approximations to order $1/c^2$'.

[5] The symbols M, m, ζ there correspond to $m, \bar{\mu}, \sigma$ in the present paper.

$$m\mathbf{a}_\parallel|_{MB} = (Q^2/4\zeta^2)2u_3\mathbf{u}_\parallel/c^2, \qquad ma_3|_{MB} = -(Q^2/4\zeta^2)\big[1 - 5u_3^2/2c^2\big], \tag{4.3}$$

differing spectacularly from (2.7).

The most obvious difference from (ii) is that MB quantizes the Maxwell field in a different gauge; by contrast, our section 2 of course features only the fields, and does not quantize at all.

(iv) As an altogether different scenario one can envisage a dissipative half-space with plasma frequency ω_p and Ohmic conductivity σ (not to be confused with the parameter σ from paragraph (ii)), and determine the drag force on an exterior charge constrained to constant $\mathbf{u}_\parallel$. For simplicity, we consider only the regime where $u/\omega_p\zeta \ll 1$. In a purely classical Maxwellian calculation focused on the B fields inside and out, Boyer (1974) found $\mathbf{f} = -\mathbf{u}Q^2/16\pi\sigma\zeta^3$. Remarkably, the same result follows from quantum-mechanical calculations which take $u/c \to 0$ from the outset (so that all the above ΔH vanish), and which ignore B fields altogether (Tomassone and Widom 1997, Barton 2000). Our point is that the mere absence of $\hbar$ from the result has indeed ensured agreement between correct calculations even when they proceed from starting points and in ways as apparently disjoint as these do.

(v) Eberlein and Robaschik (2004, 2006a, 2006b: cited as ER) report the self-energy $\Sigma(\zeta, \mathbf{p})$ of an electron quasiclassically localized outside an insulating half-space $z < 0$, with $\varepsilon = n^2$ at all frequencies. The momentum-dependent parts of Σ deliver what is effectively their expression for our ΔH. In terms of

$$\alpha \equiv (n^2 - 1)/(n^2 + 1), \qquad \lim_{n\to\infty} \alpha = 1 \tag{4.4}$$

it reads

$$H_{ER} \equiv \frac{p^2}{2m} - \frac{\alpha Q^2}{4\zeta} - \frac{p^4}{8m^3c^2} + \frac{Q^2}{m^2c^2\zeta}\left[-\left(\frac{\alpha+\alpha^2}{16}\right)p_\parallel^2 - \left(\frac{3\alpha+\alpha^2}{8}\right)p_3^2\right], \tag{4.5}$$

where one notes that

$$\lim_{n\to\infty} \Delta H|_{ER} = (Q^2/m^2c^2\zeta)\big[-p_\parallel^2/8 - p_3^2/2\big]. \tag{4.6}$$

One crucial difference from plasma models is that theirs has no dimensional inputs like ω_p or q_p, whence it has no distinct long-distance and short-distance regimes, and can approach perfect reflection only uniformly in ζ. This contrasts sharply with the plasma models[6] discussed in paragraphs (i)–(iii), where the formal perfect-reflection limit is mathematically equivalent to the long-distance limit, but at short distances becomes inoperable.

The point at issue is the apparent paradox that, even though $n \to \infty$ produces perfect reflection, (4.6) differs[7] from the perfect-reflector limit ΔH in (3.1). A classical check on (4.5) requires the image fields at the particle, and their scalar and vector potentials there; to find these one must extend the arguments of section 2 to both half-spaces, subject now to the appropriate matching conditions $E_\parallel(\mathbf{r}_\parallel, 0-) = E_\parallel(\mathbf{r}_\parallel, 0+)$ and $n^2E_3(\mathbf{r}_\parallel, 0-) = E_3(\mathbf{r}_\parallel, 0+)$. Then the outside fields determine the acceleration directly, while the outside potentials yield the effective interaction Hamiltonian. Such calculations (Barton 2007) in the nonrelativistic regime $(u/c)^2 \ll 1$ reproduce $\Delta H|_{ER}$ and the accelerations corresponding to it as long as $(nu/c)^2 \ll 1$, compatibly with very large but not with arbitrarily large n. Thus (4.6) can serve as an excellent approximation over a wide range; but the present writer thinks that as $n \to \infty$ its quantal derivation must eventually fail.

[6] Of course the plasma models are also highly restrictive, in that one and the same (and dimensional) parameter controls both the absolute magnitude and the dispersion of the reflectivity. To avoid both handicaps one would have to have to consider a dispersive insulator, say one described by $\varepsilon = (\omega_L^2 - \omega^2)/(\omega_T^2 - \omega^2)$.

[7] ER derive (4.5) via an integral over normal modes, and note that (3.1) would emerge if $n \to \infty$ were implemented in the integrand, and the integration carried out afterward. But they emphasize that the integration must be done first, yielding (4.5), and that the limit (when physically appropriate) must be implemented only on ΔH_{ER}, as in (4.6).

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